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# THE $\Lambda$ -ADIC SHIMURA-SHINTANI-WALDSPURGER CORRESPONDENCE

MATTEO LONGO, MARC-HUBERT NICOLE

ABSTRACT. We generalize the  $\Lambda$ -adic Shintani lifting for  $\mathrm{GL}_2(\mathbb{Q})$  to indefinite quaternion algebras over  $\mathbb{Q}$ .

## 1. INTRODUCTION

Langlands's principle of functoriality predicts the existence of a staggering wealth of transfers (or lifts) between automorphic forms for different reductive groups. In recent years, attempts at the formulation of  $p$ -adic variants of Langlands's functoriality have been articulated in various special cases. We prove the existence of the Shimura-Shintani-Waldspurger lift for  $p$ -adic families.

More precisely, Stevens, building on the work of Hida and Greenberg-Stevens, showed in [21] the existence of a  $\Lambda$ -adic variant of the classical Shintani lifting of [20] for  $\mathrm{GL}_2(\mathbb{Q})$ . This  $\Lambda$ -adic lifting can be seen as a formal power series with coefficients in a finite extension of the Iwasawa algebra  $\Lambda := \mathbb{Z}_p[[X]]$  equipped with specialization maps interpolating classical Shintani lifts of classical modular forms appearing in a given Hida family.

Shimura in [19], resp. Waldspurger in [22] generalized the classical Shimura-Shintani correspondence to quaternion algebras over  $\mathbb{Q}$ , resp. over any number field. In the  $p$ -adic realm, Hida ([7]) constructed a  $\Lambda$ -adic Shimura lifting, while Ramsey ([17]) (resp. Park [12]) extended the Shimura (resp. Shintani) lifting to the overconvergent setting.

In this paper, motivated by ulterior applications to Shimura curves over  $\mathbb{Q}$ , we generalize Stevens's result to any non-split rational indefinite quaternion algebra  $B$ , building on work of Shimura [19] and combining this with a result of Longo-Vigni [9]. Our main result, for which the reader is referred to Theorem 3.8 below, states the existence of a formal power series and specialization maps interpolating Shimura-Shintani-Waldspurger lifts of classical forms in a given  $p$ -adic family of automorphic forms on the quaternion algebra  $B$ . The  $\Lambda$ -adic variant of Waldspurger's result appears computationally challenging (see remark in [15, Intro.]), but it seems within reach for real quadratic fields (cf. [13]).

As an example of our main result, we consider the case of families with trivial character. Fix a prime number  $p$  and a positive integer  $N$  such that  $p \nmid N$ . Embed the set  $\mathbb{Z}^{\geq 2}$  of integers greater or equal to 2 in  $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$  by sending  $k \in \mathbb{Z}^{\geq 2}$  to the character  $x \mapsto x^{k-2}$ . Let  $f_\infty$  be an Hida family of tame level  $N$  passing through a form  $f_0$  of level  $\Gamma_0(Np)$  and weight  $k_0$ . There is a neighborhood  $U$  of  $k_0$  in  $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$  such that, for any  $k \in \mathbb{Z}^{\geq 2} \cap U$ , the weight  $k$  specialization of  $f_\infty$

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gives rise to an element  $f_k \in S_k(\Gamma_0(Np))$ . Fix a factorization  $N = MD$  with  $D > 1$  a square-free product of an even number of primes and  $(M, D) = 1$  (we assume that such a factorization exists). Applying the Jacquet-Langlands correspondence we get for any  $k \in \mathbb{Z}^{\geq 2} \cap U$  a modular form  $f_k^{\text{JL}}$  on  $\Gamma$ , which is the group of norm-one elements in an Eichler order  $R$  of level  $Mp$  contained in the indefinite rational quaternion algebra  $B$  of discriminant  $D$ . One can show that these modular forms can be  $p$ -adically interpolated, up to scaling, in a neighborhood of  $k_0$ . More precisely, let  $\mathcal{O}$  be the ring of integers of a finite extension  $F$  of  $\mathbb{Q}_p$  and let  $\mathbb{D}$  denote the  $\mathcal{O}$ -module of  $\mathcal{O}$ -valued measures on  $\mathbb{Z}_p^2$  which are supported on the set of primitive elements in  $\mathbb{Z}_p^2$ . Let  $\Gamma_0$  be the group of norm-one elements in an Eichler order  $R_0 \subseteq B$  containing  $R$ . There is a canonical action of  $\Gamma_0$  on  $\mathbb{D}$  (see [9, §2.4] for its description). Denote by  $F_k$  the extension of  $F$  generated by the Fourier coefficients of  $f_k$ . Then there is an element  $\Phi \in H^1(\Gamma_0, \mathbb{D})$  and maps

$$\rho_k : H^1(\Gamma_0, \mathbb{D}) \longrightarrow H^1(\Gamma, F_k)$$

such that  $\rho(k)(\Phi) = \phi_k$ , the cohomology class associated to  $f_k^{\text{JL}}$ , with  $k$  in a neighborhood of  $k_0$  (for this we need a suitable normalization of the cohomology class associated to  $f_k^{\text{JL}}$ , which we do not touch for simplicity in this introduction). We view  $\Phi$  as a quaternionic family of modular forms. To each  $\phi_k$  we may apply the Shimura-Shintani-Waldspurger lifting ([19]) and obtain a modular form  $h_k$  of weight  $k + 1/2$ , level  $4Np$  and trivial character. We show that this collection of forms can be  $p$ -adically interpolated. For clarity's sake, we present the liftings and their  $\Lambda$ -adic variants in a diagram, in which the horizontal maps are specialization maps of the  $p$ -adic family to weight  $k$ ; JL stands for the Jacquet-Langlands correspondence; SSW stands for the Shimura-Shintani-Waldspurger lift; and the dotted arrows are constructed in this paper:

$$\begin{array}{ccc} f_\infty & \longmapsto & f_k \\ \Lambda\text{-adic JL} \downarrow & & \downarrow \text{JL} \\ \Phi & \xrightarrow{\rho_k} & \phi_k \\ \Lambda\text{-adic SSW} \downarrow & & \downarrow \text{SSW} \\ \Theta & \dashrightarrow & h_k \end{array}$$

More precisely, as a particular case of our main result, Theorem 3.8, we get the following

**Theorem 1.1.** *There exists a  $p$ -adic neighborhood  $U_0$  of  $k_0$  in  $\text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ ,  $p$ -adic periods  $\Omega_k$  for  $k \in U_0 \cap \mathbb{Z}^{\geq 2}$  and a formal expansion*

$$\Theta = \sum_{\xi \geq 1} a_\xi q^\xi$$

*with coefficients  $a_\xi$  in the ring of  $\mathbb{C}_p$ -valued functions on  $U_0$ , such that for all  $k \in U_0 \cap \mathbb{Z}^{\geq 2}$  we have*

$$\Theta(k) = \Omega_k \cdot h_k.$$

*Further,  $\Omega_{k_0} \neq 0$ .*

## 2. SHINTANI INTEGRALS AND FOURIER COEFFICIENTS OF HALF-INTEGRAL WEIGHT MODULAR FORMS

We express the Fourier coefficients of half-integral weight modular forms in terms of period integrals, thus allowing a cohomological interpretation which is key to the production of the  $\Lambda$ -adic version of the Shimura-Shintani-Waldspurger correspondence. For the quaternionic Shimura-Shintani-Waldspurger correspondence of interest to us (see [15], [22]), the period integrals expressing the values of the Fourier coefficients have been computed generally by Prasanna in [16].

**2.1. The Shimura-Shintani-Waldspurger lifting.** Let  $4M$  be a positive integer,  $2k$  an even non-negative integer and  $\chi$  a Dirichlet character modulo  $4M$  such that  $\chi(-1) = 1$ . Recall that the space of half-integral weight modular forms  $S_{k+1/2}(4M, \chi)$  consists of holomorphic cuspidal functions  $h$  on the upper-half plane  $\mathfrak{H}$  such that

$$h(\gamma(z)) = j^{1/2}(\gamma, z)^{2k+1} \chi(d) h(z),$$

for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4M)$ , where  $j^{1/2}(\gamma, z)$  is the standard square root of the usual automorphy factor  $j(\gamma, z)$  (cf. [15, 2.3]).

To any quaternionic integral weight modular form we may associate a half-integral weight modular form following Shimura's work [19], as we will describe below.

Fix an odd square free integer  $N$  and a factorization  $N = M \cdot D$  into coprime integers such that  $D > 1$  is a product of an even number of distinct primes. Fix a Dirichlet character  $\psi$  modulo  $M$  and a positive even integer  $2k$ . Suppose that

$$\psi(-1) = (-1)^k.$$

Define the Dirichlet character  $\chi$  modulo  $4N$  by

$$\chi(x) := \psi(x) \left( \frac{-1}{x} \right)^k.$$

Let  $B$  be an indefinite quaternion algebra over  $\mathbb{Q}$  of discriminant  $D$ . Fix a maximal order  $\mathcal{O}_B$  of  $B$ . For every prime  $\ell \nmid M$ , choose an isomorphism

$$i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathbb{M}_2(\mathbb{Q}_\ell)$$

such that  $i_\ell(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) = \mathbb{M}_2(\mathbb{Z}_\ell)$ . Let  $R \subseteq \mathcal{O}_B$  be the Eichler order of  $B$  of level  $M$  defined by requiring that  $i_\ell(R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$  is the suborder of  $\mathbb{M}_2(\mathbb{Z}_\ell)$  of upper triangular matrices modulo  $\ell$  for all  $\ell \nmid M$ . Let  $\Gamma$  denote the subgroup of the group  $R_1^\times$  of norm 1 elements in  $R^\times$  consisting of those  $\gamma$  such that  $i_\ell(\gamma) \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\ell}$  for all  $\ell \nmid M$ . We denote by  $S_{2k}(\Gamma)$  the  $\mathbb{C}$ -vector space of weight  $2k$  modular forms on  $\Gamma$ , and by  $S_{2k}(\Gamma, \psi^2)$  the subspace of  $S_{2k}(\Gamma)$  consisting of forms having character  $\psi^2$  under the action of  $R_1^\times$ . Fix a Hecke eigenform

$$f \in S_{2k}(\Gamma, \psi^2)$$

as in [19, Section 3].

Let  $V$  denote the  $\mathbb{Q}$ -subspace of  $B$  consisting of elements with trace equal to zero. For any  $v \in V$ , which we view as a trace zero matrix in  $\mathbb{M}_2(\mathbb{R})$  (after fixing an isomorphism  $i_\infty : B \otimes \mathbb{R} \simeq \mathbb{M}_2(\mathbb{R})$ ), set

$$G_v := \{\gamma \in \mathrm{SL}_2(\mathbb{R}) \mid \gamma^{-1} v \gamma = v\}$$

and put  $\Gamma_v := G_v \cap \Gamma$ . One can show that there exists an isomorphism  $\omega : \mathbb{R}^\times \xrightarrow{\sim} G_v$  defined by  $\omega(s) = \beta^{-1} \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \beta$ , for some  $\beta \in \mathrm{SL}_2(\mathbb{R})$ . Let  $\mathfrak{t}_v$  be the order of  $\Gamma_v \cap \{\pm 1\}$  and let  $\gamma_v$  be an element of  $\Gamma_v$  which generates  $\Gamma_v \setminus \{\pm 1\}$ . Changing  $\gamma_v$  to  $\gamma_v^{-1}$  if necessary, we may assume  $\gamma_v = \omega(t)$  with  $t > 0$ . Define  $V^*$  to be the  $\mathbb{Q}$ -subspace of  $V$  consisting of elements with strictly negative norm. For any  $\alpha = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in V^*$  and  $z \in \mathcal{H}$ , define the quadratic form

$$Q_\alpha(z) := cz^2 - 2az - b.$$

Fix  $\tau \in \mathcal{H}$  and set

$$P(f, \alpha, \Gamma) := -(2(-\mathrm{nr}(\alpha))^{1/2}/\mathfrak{t}_\alpha) \int_\tau^{\gamma_\alpha(\tau)} Q_\alpha(z)^{k-1} f(z) dz$$

where  $\mathrm{nr} : B \rightarrow \mathbb{Q}$  is the norm map. By [19, Lemma 2.1], the integral is independent on the choice  $\tau$ , which justifies the notation.

*Remark 2.1.* The definition of  $P(f, \alpha, \Gamma)$  given in [19, (2.5)] looks different: the above expression can be derived as in [19, page 629] by means of [19, (2.20) and (2.22)].

Let  $R(\Gamma)$  denote the set of equivalence classes of  $V^*$  under the action of  $\Gamma$  by conjugation. By [19, (2.6)],  $P(f, \alpha, \Gamma)$  only depends on the conjugacy class of  $\alpha$ , and thus, for  $\mathcal{C} \in R(\Gamma)$ , we may define  $P(f, \mathcal{C}, \Gamma) := P(f, \alpha, \Gamma)$  for any choice of  $\alpha \in \mathcal{C}$ . Also,  $q(\mathcal{C}) := -\mathrm{nr}(\alpha)$  for any  $\alpha \in \mathcal{C}$ .

Define  $\mathcal{O}'_B$  to be the maximal order in  $B$  such that  $\mathcal{O}'_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simeq \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$  for all  $\ell \nmid M$  and  $\mathcal{O}'_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$  is equal to the local order of  $B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$  consisting of elements  $\gamma$  such that  $i_\ell(\gamma) = \begin{pmatrix} a & b/M \\ cM & d \end{pmatrix}$  with  $a, b, c, d \in \mathbb{Z}_\ell$ , for all  $\ell \nmid M$ . Given  $\alpha \in \mathcal{O}'_B$ , we can find an integer  $b_\alpha$  such that

$$(1) \quad i_\ell(\alpha) \equiv \begin{pmatrix} * & b_\alpha/M \\ * & * \end{pmatrix} \pmod{i_\ell(R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)}, \quad \forall \ell \nmid M.$$

Define a locally constant function  $\eta_\psi$  on  $V$  by  $\eta_\psi(\alpha) = \psi(b_\alpha)$  if  $\alpha \in \mathcal{O}'_B \cap V$  and  $\eta(\alpha) = 0$  otherwise, with  $\psi(a) = 0$  if  $(a, M) \neq 1$  (for the definition of locally constant functions on  $V$  in this context, we refer to [19, p. 611]).

For any  $\mathcal{C} \in R(\Gamma)$ , fix  $\alpha_{\mathcal{C}} \in \mathcal{C}$ . For any integer  $\xi \geq 1$ , define

$$a_\xi(\tilde{h}) := (2\mu(\Gamma \backslash \mathfrak{H}))^{-1} \cdot \sum_{\mathcal{C} \in R(\Gamma), q(\mathcal{C})=\xi} \eta_\psi(\alpha_{\mathcal{C}}) \xi^{-1/2} P(f, \mathcal{C}, \Gamma).$$

Then, by [19, Theorem 3.1],

$$\tilde{h} := \sum_{\xi \geq 1} a_\xi(\tilde{h}) q^\xi \in S_{k+1/2}(4N, \chi)$$

is called the Shimura-Shintani-Waldspurger lifting of  $f$ .

**2.2. Cohomological interpretation.** We introduce necessary notation to define the action of the Hecke action on cohomology groups; for details, see [9, §2.1]. If  $G$  is a subgroup of  $B^\times$  and  $S$  a subsemigroup of  $B^\times$  such that  $(G, S)$  is an Hecke pair, we let  $\mathcal{H}(G, S)$  denote the Hecke algebra corresponding to  $(G, S)$ , whose elements are written as  $T(s) = GsG = \coprod_i Gs_i$  for  $s, s_i \in S$  (finite disjoint union). For any  $s \in S$ , let  $s^* := \mathrm{norm}(s)s^{-1}$  and denote by  $S^*$  the set of elements of the form  $s^*$  for  $s \in S$ . For any  $\mathbb{Z}[S^*]$ -module  $M$  we let  $T(s)$  act on  $H^1(G, M)$  at the level of cochains  $c \in Z^1(G, M)$  by the formula  $(c|T(s))(\gamma) = \sum_i s_i^* c(t_i(\gamma))$ , where  $t_i(\gamma)$  are

defined by the equations  $Gs_i\gamma = Gs_j$  and  $s_i\gamma = t_i(\gamma)s_j$ . In the following, we will consider the case of  $G = \Gamma$  and

$$S = \{s \in B^\times \mid i_\ell(s) \text{ is congruent to } \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{\ell} \text{ for all } \ell \mid M\}.$$

For any field  $L$  and any integer  $n \geq 0$ , let  $V_n(L)$  denote the  $L$ -dual of the  $L$ -vector space  $\mathcal{P}_n(L)$  of homogeneous polynomials in 2 variables of degree  $n$ . We let  $\mathbb{M}_2(L)$  act from the right on  $P(x, y)$  as  $P|\gamma(x, y) := P(\gamma(x, y))$ , where for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  we put

$$\gamma(x, y) := (ax + yb, cx + dy).$$

This also equips  $V_n(L)$  with a left action by  $\gamma \cdot \varphi(P) := \varphi(P|\gamma)$ . To simplify the notation, we will write  $P(z)$  for  $P(z, 1)$ .

Let  $F$  denote the finite extension of  $\mathbb{Q}$  generated by the eigenvalues of the Hecke action on  $f$ . For any field  $K$  containing  $F$ , set

$$\mathbb{W}_f(K) := H^1(\Gamma, V_{k-2}(K))^f$$

where the superscript  $f$  denotes the subspace on which the Hecke algebra acts via the character associated with  $f$ . Also, for any sign  $\pm$ , let  $\mathbb{W}_f^\pm(K)$  denote the  $\pm$ -eigenspace for the action of the archimedean involution  $\iota$ . Remember that  $\iota$  is defined by choosing an element  $\omega_\infty$  of norm  $-1$  in  $R^\times$  such that  $i_\ell(\omega_\infty) \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pmod{M}$  for all primes  $\ell \mid M$  and then setting  $\iota := T(\omega_\infty)$  (see [9, §2.1]). Then  $\mathbb{W}_f^\pm(K)$  is one dimensional (see, e.g., [9, Proposition 2.2]); fix a generator  $\phi_f^\pm$  of  $\mathbb{W}_f^\pm(F)$ .

To explicitly describe  $\phi_f^\pm$ , let us introduce some more notation. Define

$$f|\omega_\infty(z) := (Cz + D)^{-k/2} \overline{f(\omega_\infty(\bar{z}))}$$

where  $i_\infty(\omega_\infty) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ . Then  $f|\omega_\infty \in S_{2k}(\Gamma)$  as well. If the eigenvalues of the Hecke action on  $f$  are real, then we may assume, after multiplying  $f$  by a scalar, that  $f|\omega_\infty = f$  (see [19, p. 627] or [10, Lemma 4.15]). In general, let  $I(f)$  denote the class in  $H^1(\Gamma, V_{k-2}(\mathbb{C}))$  represented by the cocycle

$$\gamma \mapsto \left[ P \mapsto I_\gamma(f)(P) := \int_\tau^{\gamma(\tau)} f(z)P(z)dz \right]$$

for any  $\tau \in \mathcal{H}$  (the corresponding class is independent on the choice of  $\tau$ ). With this notation,

$$P(f, \alpha, \Gamma) = -(2(-\text{nr}(\alpha))^{1/2}/\mathfrak{t}_\alpha) \cdot I_{\gamma_{\alpha_C}}(f)(Q_{\alpha_C}(z)^{k-1}).$$

Denote by  $I^\pm(f) := (1/2) \cdot I(f) \pm (1/2) \cdot I(f)|\omega_\infty$ , the projection of  $I(f)$  to the eigenspaces for the action of  $\omega_\infty$ . Then  $I(f) = I^+(f) + I^-(f)$  and  $I_f^\pm = \Omega_f^\pm \cdot \phi_f^\pm$ , for some  $\Omega_f^\pm \in \mathbb{C}^\times$ .

Given  $\alpha \in V^*$  of norm  $-\xi$ , put  $\alpha' := \omega_\infty^{-1}\alpha\omega_\infty$ . By [19, 4.19], we have

$$\eta(\alpha)\xi^{-1/2}P(f, \alpha, \Gamma) + \eta(\alpha')\xi^{-1/2}P(f, \alpha', \Gamma) = -\eta(\alpha) \cdot \mathfrak{t}_\alpha^{-1} \cdot I_{\gamma_\alpha}^+(Q_{\alpha_C}(z)^{k-1}).$$

We then have

$$a_\xi(\tilde{h}) = \sum_{C \in R_2(\Gamma), q(C)=\xi} \frac{-\eta_\psi(\alpha_C)}{2\mu(\Gamma \backslash \mathcal{H}) \cdot \mathfrak{t}_{\alpha_C}} \cdot I_{\gamma_{\alpha_C}}^+(Q_{\alpha_C}(z)^{k-1}).$$

We close this section by choosing a suitable multiple of  $h$  which will be the object of the next section. Given  $Q_\alpha(z) = cz^2 - 2az - b$  as above, with  $\alpha$  in

$V^*$ , define  $\tilde{Q}_\alpha(z) := M \cdot Q_\alpha(z)$ . Then, clearly,  $I^\pm(f)(\tilde{Q}_{\alpha_c}(z)^{k-1})$  is equal to  $M^{k-1}I^\pm(f)(Q_{\alpha_c}(z)^{k-1})$ . We thus normalize the Fourier coefficients by setting

$$(2) \quad a_\xi(h) := -\frac{a_\xi(\tilde{h}) \cdot M^{k-1} \cdot 2\mu(\Gamma \backslash \mathcal{H})}{\Omega_f^+} = \sum_{C \in R(\Gamma), q(C)=\xi} \frac{\eta_\psi(\alpha_C)}{\mathfrak{t}_{\alpha_C}} \cdot \phi_f^+(\tilde{Q}_{\alpha_C}(z)^{k-1}).$$

So

$$(3) \quad h := \sum_{\xi \geq 1} a_\xi(h) q^\xi$$

belongs to  $S_{k+1/2}(4N, \chi)$  and is a non-zero multiple of  $\tilde{h}$ .

### 3. THE $\Lambda$ -ADIC SHIMURA-SHINTANI-WALDSPURGER CORRESPONDENCE

At the heart of Stevens's proof lies the control theorem of Greenberg-Stevens, which has been worked out in the quaternionic setting by Longo-Vigni [9].

Recall that  $N \geq 1$  is a square free integer and fix a decomposition  $N = M \cdot D$  where  $D$  is a square free product of an even number of primes and  $M$  is coprime to  $D$ . Let  $p \nmid N$  be a prime number and fix an embedding  $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ .

**3.1. The Hida Hecke algebra.** Fix an ordinary  $p$ -stabilized newform

$$(4) \quad f_0 \in S_{k_0}(\Gamma_1(Mp^{r_0}) \cap \Gamma_0(D), \epsilon_0)$$

of level  $\Gamma_1(Mp^{r_0}) \cap \Gamma_0(D)$ , Dirichlet character  $\epsilon_0$  and weight  $k_0$ , and write  $\mathcal{O}$  for the ring of integers of the field generated over  $\mathbb{Q}_p$  by the Fourier coefficients of  $f_0$ .

Let  $\Lambda$  (respectively,  $\mathcal{O}[[\mathbb{Z}_p^\times]]$ ) denote the Iwasawa algebra of  $W := 1 + p\mathbb{Z}_p$  (respectively,  $\mathbb{Z}_p^\times$ ) with coefficients in  $\mathcal{O}$ . We denote group-like elements in  $\Lambda$  and  $\mathcal{O}[[\mathbb{Z}_p^\times]]$  as  $[t]$ . Let  $\mathfrak{h}_\infty^{\text{ord}}$  denote the  $p$ -ordinary Hida Hecke algebra with coefficients in  $\mathcal{O}$  of tame level  $\Gamma_1(N)$ . Denote by  $\mathcal{L} := \text{Frac}(\Lambda)$  the fraction field of  $\Lambda$ . Let  $\mathcal{R}$  denote the integral closure of  $\Lambda$  in the primitive component  $\mathcal{K}$  of  $\mathfrak{h}_\infty^{\text{ord}} \otimes_\Lambda \mathcal{L}$  corresponding to  $f_0$ . It is well known that the  $\Lambda$ -algebra  $\mathcal{R}$  is finitely generated as  $\Lambda$ -module.

Denote by  $\mathcal{X}$  the  $\mathcal{O}$ -module  $\text{Hom}_{\mathcal{O}\text{-alg}}^{\text{cont}}(\mathcal{R}, \bar{\mathbb{Q}}_p)$  of continuous homomorphisms of  $\mathcal{O}$ -algebras. Let  $\mathcal{X}^{\text{arith}}$  the set of arithmetic homomorphisms in  $\mathcal{X}$ , defined in [9, §2.2] by requiring that the composition

$$W \hookrightarrow \Lambda \xrightarrow{\kappa} \bar{\mathbb{Q}}_p$$

has the form  $\gamma \mapsto \psi_\kappa(\gamma)\gamma^{n_\kappa}$  with  $n_\kappa = k_\kappa - 2$  for an integer  $k_\kappa \geq 2$  (called the weight of  $\kappa$ ) and a finite order character  $\psi_\kappa : W \rightarrow \bar{\mathbb{Q}}_p$  (called the wild character of  $\kappa$ ). Denote by  $r_\kappa$  the smallest among the positive integers  $t$  such that  $1 + p^t\mathbb{Z}_p \subseteq \ker(\psi_\kappa)$ . For any  $\kappa \in \mathcal{X}^{\text{arith}}$ , let  $P_\kappa$  denote the kernel of  $\kappa$  and  $\mathcal{R}_{P_\kappa}$  the localization of  $\mathcal{R}$  at  $\kappa$ . The field  $F_\kappa := \mathcal{R}_{P_\kappa}/P_\kappa\mathcal{R}_{P_\kappa}$  is a finite extension of  $\text{Frac}(\mathcal{O})$ . Further, by duality,  $\kappa$  corresponds to a normalized eigenform

$$f_\kappa \in S_{k_\kappa}(\Gamma_0(Np^{r_\kappa}), \epsilon_\kappa)$$

for a Dirichlet character  $\epsilon_\kappa : (\mathbb{Z}/Np^{r_\kappa}\mathbb{Z})^\times \rightarrow \bar{\mathbb{Q}}_p$ . More precisely, if we write  $\psi_\mathcal{R}$  for the character of  $\mathcal{R}$ , defined as in [6, Terminology p. 555], and we let  $\omega$  denote the Teichmüller character, we have  $\epsilon_\kappa := \psi_\kappa \cdot \psi_\mathcal{R} \cdot \omega^{-n_\kappa}$  (see [6, Cor. 1.6]). We call  $(\epsilon_\kappa, k_\kappa)$  the signature of  $\kappa$ . We let  $\kappa_0$  denote the arithmetic character associated with  $f_0$ , so  $f_0 = f_{\kappa_0}$ ,  $k_0 = k_{\kappa_0}$ ,  $\epsilon_0 = \epsilon_{\kappa_0}$ , and  $r_0 = r_{\kappa_0}$ . The eigenvalues of  $f_\kappa$  under the action of the Hecke operators  $T_n$  ( $n \geq 1$  an integer) belong to  $F_\kappa$ . Actually, one can show that  $f_\kappa$  is a  $p$ -stabilized newform on  $\Gamma_1(Mp^{r_\kappa}) \cap \Gamma_0(D)$ .

Let  $\Lambda_N$  denote the Iwasawa algebra of  $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$  with coefficients in  $\mathcal{O}$ . To simplify the notation, define  $\Delta := (\mathbb{Z}/Np\mathbb{Z})^\times$ . We have a canonical isomorphism of rings  $\Lambda_N \simeq \Lambda[\Delta]$ , which makes  $\Lambda_N$  a  $\Lambda$ -algebra, finitely generated as  $\Lambda$ -module. Define the tensor product of  $\Lambda$ -algebras

$$\mathcal{R}_N := \mathcal{R} \otimes_\Lambda \Lambda_N,$$

which is again a  $\Lambda$ -algebra (resp.  $\Lambda_N$ -algebra) finitely generated as a  $\Lambda$ -module, (resp. as a  $\Lambda_N$ -module). One easily checks that there is a canonical isomorphism of  $\Lambda$ -algebras

$$\mathcal{R}_N \simeq \mathcal{R}[\Delta]$$

(where  $\Lambda$  acts on  $\mathcal{R}$ ); this is also an isomorphism of  $\Lambda_N$ -algebras, when we let  $\Lambda_N \simeq \Lambda[\Delta]$  act on  $\mathcal{R}[\Delta]$  in the obvious way.

We can extend any  $\kappa \in \mathcal{X}^{\text{arith}}$  to a continuous  $\mathcal{O}$ -algebra morphism

$$\kappa_N : \mathcal{R}_N \longrightarrow \bar{\mathbb{Q}}_p$$

setting

$$\kappa_N \left( \sum_{i=1}^n r_i \cdot \delta_i \right) := \sum_{i=1}^n \kappa(r_i) \cdot \psi_{\mathcal{R}}(\delta_i)$$

for  $r_i \in \mathcal{R}$  and  $\delta_i \in \Delta$ . Therefore,  $\kappa_N$  restricted to  $\mathbb{Z}_p^\times$  is the character  $t \mapsto \epsilon_\kappa(t)t^{n_\kappa}$ . If we denote by  $\mathcal{X}_N$  the  $\mathcal{O}$ -module of continuous  $\mathcal{O}$ -algebra homomorphisms from  $\mathcal{R}_N$  to  $\bar{\mathbb{Q}}_p$ , the above correspondence sets up an injective map  $\mathcal{X}^{\text{arith}} \hookrightarrow \mathcal{X}_N$ . Let  $\mathcal{X}_N^{\text{arith}}$  denote the image of  $\mathcal{X}^{\text{arith}}$  under this map. For  $\kappa_N \in \mathcal{X}_N^{\text{arith}}$ , we define the signature of  $\kappa_N$  to be that of the corresponding  $\kappa$ .

**3.2. The control theorem in the quaternionic setting.** Recall that  $B/\mathbb{Q}$  is a quaternion algebra of discriminant  $D$ . Fix an auxiliary real quadratic field  $F$  such that all primes dividing  $D$  are inert in  $F$  and all primes dividing  $Mp$  are split in  $F$ , and an isomorphism  $i_F : B \otimes_{\mathbb{Q}} F \simeq \mathbb{M}_2(F)$ . Let  $\mathcal{O}_B$  denote the maximal order of  $B$  obtained by taking the intersection of  $B$  with  $\mathbb{M}_2(\mathcal{O}_F)$ , where  $\mathcal{O}_F$  is the ring of integers of  $F$ . More precisely, define

$$\mathcal{O}_B := \iota^{-1}(i_F^{-1}(i_F(B \otimes 1) \cap \mathbb{M}_2(\mathcal{O}_F)))$$

where  $\iota : B \hookrightarrow B \otimes_{\mathbb{Q}} F$  is the inclusion defined by  $b \mapsto b \otimes 1$ . This is a maximal order in  $B$  because  $i_F(B \otimes 1) \cap \mathbb{M}_2(\mathcal{O}_F)$  is a maximal order in  $i_F(B \otimes 1)$ . In particular,  $i_F$  and our fixed embedding of  $\mathbb{Q}$  into  $\bar{\mathbb{Q}}_p$  induce an isomorphism

$$i_p : B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{M}_2(\mathbb{Q}_p)$$

such that  $i_p(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \mathbb{M}_2(\mathbb{Z}_p)$ . For any prime  $\ell|M$ , also choose an embedding  $\mathbb{Q} \hookrightarrow \bar{\mathbb{Q}}_\ell$  which, composed with  $i_F$ , yields isomorphisms

$$i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathbb{M}_2(\mathbb{Q}_\ell)$$

such that  $i_p(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) = \mathbb{M}_2(\mathbb{Z}_\ell)$ . Define an Eichler order  $R \subseteq \mathcal{O}_B$  of level  $M$  by requiring that for all primes  $\ell|M$  the image of  $R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$  via  $i_\ell$  consists of upper triangular matrices modulo  $\ell$ . For any  $r \geq 0$ , let  $\Gamma_r$  denote the subgroup of the group  $R_1^\times$  of norm-one elements in  $R$  consisting of those  $\gamma$  such that  $i_\ell(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $c \equiv 0 \pmod{Mp^r}$  and  $a \equiv d \equiv 1 \pmod{Mp^r}$ , for all primes  $\ell|Mp$ . To conclude this list of notation and definitions, fix an embedding  $F \hookrightarrow \mathbb{R}$  and let

$$i_\infty : B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{M}_2(\mathbb{R})$$

be the induced isomorphism.



Let  $\mathbb{Y} := \mathbb{Z}_p^2$  and denote by  $\mathbb{X}$  the set of primitive vectors in  $\mathbb{Y}$ . Let  $\mathbb{D}$  denote the  $\mathcal{O}$ -module of  $\mathcal{O}$ -valued measures on  $\mathbb{Y}$  which are supported on  $\mathbb{X}$ . Note that  $\mathbb{M}_2(\mathbb{Z}_p)$  acts on  $\mathbb{Y}$  by left multiplication; this induces an action of  $\mathbb{M}_2(\mathbb{Z}_p)$  on the  $\mathcal{O}$ -module of  $\mathcal{O}$ -valued measures on  $\mathbb{Y}$ , which induces an action on  $\mathbb{D}$ . The group  $R^\times$  acts on  $\mathbb{D}$  via  $i_p$ . In particular, we may define the group:

$$\mathbb{W} := H^1(\Gamma_0, \mathbb{D}).$$

Then  $\mathbb{D}$  has a canonical structure of  $\mathcal{O}[[\mathbb{Z}_p^\times]]$ -module, as well as  $\mathfrak{h}_\infty^{\text{ord}}$ -action, as described in [9, §2.4]. In particular, let us recall that, for any  $[t] \in \mathcal{O}[[\mathbb{Z}_p^\times]]$ , we have

$$\int_{\mathbb{X}} \varphi(x, y) d([t] \cdot \nu) = \int_{\mathbb{X}} \varphi(tx, ty) d\nu,$$

for any locally constant function  $\varphi$  on  $\mathbb{X}$ .

For any  $\kappa \in \mathcal{X}^{\text{arith}}$  and any sign  $\pm \in \{-, +\}$ , set

$$\mathbb{W}_\kappa^\pm := \mathbb{W}_{f_\kappa^{\text{JL}}}^\pm(F_\kappa) = H^1(\Gamma_{r_\kappa}, V_{n_\kappa}(F_\kappa))^{f_\kappa, \pm}$$

where  $f_\kappa^{\text{JL}}$  is any Jacquet-Langlands lift of  $f_\kappa$  to  $\Gamma_{r_\kappa}$ ; recall that the superscript  $f_\kappa$  denotes the subspace on which the Hecke algebra acts via the character associated with  $f_\kappa$ , and the superscript  $\pm$  denotes the  $\pm$ -eigenspace for the action of the archimedean involution  $\iota$ . Also, recall that  $\mathbb{W}_\kappa^\pm$  is one dimensional and fix a generator  $\phi_\kappa^\pm$  of it.

We may define specialization maps

$$\rho_\kappa : \mathbb{D} \longrightarrow V_{n_\kappa}(F_\kappa)$$

by the formula

$$(5) \quad \rho_\kappa(\nu)(P) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon_\kappa(y) P(x, y) d\nu$$

which induces (see [9, §2.5]) a map:

$$\rho_\kappa : \mathbb{W}^{\text{ord}} \longrightarrow \mathbb{W}_\kappa^{\text{ord}}.$$

Here  $\mathbb{W}^{\text{ord}}$  and  $\mathbb{W}_\kappa^{\text{ord}}$  denote the ordinary submodules of  $\mathbb{W}$  and  $\mathbb{W}_\kappa$ , respectively, defined as in [3, Definition 2.2] (see also [9, §3.5]). We also let  $\mathbb{W}_{\mathcal{R}} := \mathbb{W} \otimes_\Lambda \mathcal{R}$ , and extend the above map  $\rho_\kappa$  to a map

$$\rho_\kappa : \mathbb{W}_{\mathcal{R}}^{\text{ord}} \longrightarrow \mathbb{W}_\kappa^{\text{ord}}$$

by setting  $\rho_\kappa(x \otimes r) := \rho_\kappa(x) \cdot \kappa(r)$ .

**Theorem 3.1.** *There exists a  $p$ -adic neighborhood  $\mathcal{U}_0$  of  $\kappa_0$  in  $\mathcal{X}$ , elements  $\Phi^\pm$  in  $\mathbb{W}_{\mathcal{R}}^{\text{ord}}$  and choices of  $p$ -adic periods  $\Omega_\kappa^\pm \in F_\kappa$  for  $\kappa \in \mathcal{U}_0 \cap \mathcal{X}^{\text{arith}}$  such that, for all  $\kappa \in \mathcal{U}_0 \cap \mathcal{X}^{\text{arith}}$ , we have*

$$\rho_\kappa(\Phi^\pm) = \Omega_\kappa^\pm \cdot \phi_\kappa^\pm$$

and  $\Omega_{\kappa_0}^\pm \neq 0$ .

*Proof.* This is an easy consequence of [9, Theorem 2.18] and follows along the lines of the proof of [21, Theorem 5.5], cf. [10, Proposition 3.2].  $\square$

We now normalize our choices as follows. With  $\mathcal{U}_0$  as above, define

$$\mathcal{U}_0^{\text{arith}} := \mathcal{U}_0 \cap \mathcal{X}^{\text{arith}}.$$

Fix  $\kappa \in \mathcal{U}_0^{\text{arith}}$  and an embedding  $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ . Let  $f_\kappa^{\text{JL}}$  denote a modular form on  $\Gamma_{r_\kappa}$  corresponding to  $f_\kappa$  by the Jacquet-Langlands correspondence, which is well

defined up to elements in  $\mathbb{C}^\times$ . View  $\phi_\kappa^\pm$  as an element in  $H^1(\Gamma_{r_\kappa}, V_n(\mathbb{C}))^\pm$ . Choose a representative  $\Phi_\gamma^\pm$  of  $\Phi^\pm$ , by which we mean that if  $\Phi^\pm = \sum_i \Phi_i^\pm \otimes r_i$ , then we choose a representative  $\Phi_{i,\gamma}^\pm$  for all  $i$ . Also, we will write  $\rho_\kappa(\Phi)(P)$  as

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon_\kappa(y) P(x, y) d\Phi_\gamma^\pm := \sum_i \kappa(r_i) \cdot \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon_\kappa(y) P(x, y) d\Phi_{i,\gamma}^\pm.$$

With this notation, we see that the two cohomology classes

$$\gamma \mapsto \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon_\kappa(y) P(x, y) d\Phi_\gamma^\pm(x, y)$$

and

$$\gamma \mapsto \Omega_\kappa^\pm \cdot \int_\tau^{\gamma(\tau)} P(z, 1) f_\kappa^{\text{JL}, \pm}(z) dz$$

are cohomologous in  $H^1(\Gamma_{r_\kappa}, V_{n_\kappa}(\mathbb{C}))$ , for any choice of  $\tau \in \mathcal{H}$ .

**3.3. Metaplectic Hida Hecke algebras.** Let  $\sigma : \Lambda_N \rightarrow \Lambda_N$  be the ring homomorphism associated to the group homomorphism  $t \mapsto t^2$  on  $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ , and denote by the same symbol its restriction to  $\Lambda$  and  $\mathcal{O}[\mathbb{Z}_p^\times]$ . We let  $\Lambda_\sigma$ ,  $\mathcal{O}[\mathbb{Z}_p^\times]_\sigma$  and  $\Lambda_{N,\sigma}$  denote, respectively,  $\Lambda$ ,  $\mathcal{O}[\mathbb{Z}_p^\times]$  and  $\Lambda_N$  viewed as algebras over themselves via  $\sigma$ . The ordinary metaplectic  $p$ -adic Hida Hecke algebra we will consider is the  $\Lambda$ -algebra

$$\tilde{\mathcal{R}} := \mathcal{R} \otimes_\Lambda \Lambda_\sigma.$$

Define as above

$$\tilde{\mathcal{X}} := \text{Hom}_{\mathcal{O}\text{-alg}}^{\text{cont}}(\tilde{\mathcal{R}}, \bar{\mathbb{Q}}_p)$$

and let the set  $\tilde{\mathcal{X}}^{\text{arith}}$  of arithmetic points in  $\tilde{\mathcal{X}}$  to consist of those  $\tilde{\kappa}$  such that the composition

$$W \hookrightarrow \Lambda \xrightarrow{\lambda \mapsto 1 \otimes \lambda} \tilde{\mathcal{R}} \xrightarrow{\tilde{\kappa}} \bar{\mathbb{Q}}_p$$

has the form  $\gamma \mapsto \psi_{\tilde{\kappa}}(\gamma) \gamma^{n_{\tilde{\kappa}}}$  with  $n_{\tilde{\kappa}} := k_{\tilde{\kappa}} - 2$  for an integer  $k_{\tilde{\kappa}} \geq 2$  (called the weight of  $\tilde{\kappa}$ ) and a finite order character  $\psi_{\tilde{\kappa}} : W \rightarrow \bar{\mathbb{Q}}$  (called the wild character of  $\tilde{\kappa}$ ). Let  $r_{\tilde{\kappa}}$  the smallest among the positive integers  $t$  such that  $1 + p^t \mathbb{Z}_p \subseteq \ker(\psi_{\tilde{\kappa}})$ .

We have a map  $p : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  induced by pull-back from the canonical map  $\mathcal{R} \rightarrow \tilde{\mathcal{R}}$ . The map  $p$  restricts to arithmetic points.

As above, define the  $\Lambda$ -algebra (or  $\Lambda_N$ -algebra)

$$(6) \quad \tilde{\mathcal{R}}_N := \mathcal{R} \otimes_\Lambda \Lambda_{N,\sigma}$$

via  $\lambda \mapsto 1 \otimes \lambda$ .

We easily see that

$$\tilde{\mathcal{R}}_N \simeq \tilde{\mathcal{R}}[\Delta]$$

as  $\Lambda_N$ -algebras, where we enhance  $\tilde{\mathcal{R}}[\Delta]$  with the following structure of  $\Lambda_N \simeq \Lambda[\Delta]$ -algebra: for  $\sum_i \lambda_i \cdot \delta_i \in \Lambda[\Delta]$  (with  $\lambda_i \in \Lambda$  and  $\delta_i \in \Delta$ ) and  $\sum_j r_j \cdot \delta'_j \in \tilde{\mathcal{R}}[\Delta]$  (with  $r_j = \sum_h r_{j,h} \otimes \lambda_{j,h} \in \tilde{\mathcal{R}}$ ,  $r_{j,h} \in \mathcal{R}$ ,  $\lambda_{j,h} \in \Lambda_\sigma$ , and  $\delta'_j \in \Delta$ ), we set

$$\left( \sum_i \lambda_i \cdot \delta_i \right) \cdot \left( \sum_j r_j \cdot \delta'_j \right) := \sum_{i,j,h} (r_{j,h} \otimes (\lambda_i \lambda_{j,h})) \cdot (\delta_i \delta'_j).$$

As above, extend  $\tilde{\kappa} \in \tilde{\mathcal{X}}^{\text{arith}}$  to a continuous  $\mathcal{O}$ -algebra morphism  $\tilde{\kappa}_N : \tilde{\mathcal{R}}_N \rightarrow \bar{\mathbb{Q}}_p$  by setting

$$\tilde{\kappa}_N \left( \sum_{i=1}^n x_i \cdot \delta_i \right) := \sum_{i=1}^n \tilde{\kappa}(x_i) \cdot \psi_{\mathcal{R}}(\delta_i)$$

for  $x_i \in \tilde{\mathcal{R}}$  and  $\delta_i \in \Delta$ , where  $\psi_{\mathcal{R}}$  is the character of  $\mathcal{R}$ . If we denote by  $\tilde{\mathcal{X}}_N$  the  $\mathcal{O}$ -module of continuous  $\mathcal{O}$ -linear homomorphisms from  $\tilde{\mathcal{R}}_N$  to  $\bar{\mathbb{Q}}_p$ , the above correspondence sets up an injective map  $\tilde{\mathcal{X}}^{\text{arith}} \hookrightarrow \tilde{\mathcal{X}}_N$  and we let  $\tilde{\mathcal{X}}_N^{\text{arith}}$  denote the image of  $\tilde{\mathcal{X}}^{\text{arith}}$ . Put  $\epsilon_{\tilde{\kappa}} := \psi_{\tilde{\kappa}} \cdot \psi_{\mathcal{R}} \cdot \omega^{-n_{\tilde{\kappa}}}$ , which we view as a Dirichlet character of  $(\mathbb{Z}/Np^{r_{\tilde{\kappa}}}\mathbb{Z})^\times$ , and call the pair  $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$  the signature of  $\tilde{\kappa}_N$ , where  $\tilde{\kappa}$  is the arithmetic point corresponding to  $\tilde{\kappa}_N$ .

We also have a map  $p_N : \tilde{\mathcal{X}}_N \rightarrow \mathcal{X}_N$  induced from the map  $\mathcal{R}_N \rightarrow \tilde{\mathcal{R}}_N$  taking  $r \mapsto r \otimes 1$  by pull-back. The map  $p_N$  also restricts to arithmetic points. The maps  $p$  and  $p_N$  make the following diagram commute:

$$\begin{array}{ccc} \tilde{\mathcal{X}}^{\text{arith}} & \hookrightarrow & \tilde{\mathcal{X}}_N^{\text{arith}} \\ \downarrow p & & \downarrow p_N \\ \mathcal{X}^{\text{arith}} & \hookrightarrow & \mathcal{X}_N^{\text{arith}} \end{array}$$

where the projections take a signature  $(\epsilon, k)$  to  $(\epsilon^2, 2k)$ .

**3.4. The  $\Lambda$ -adic correspondence.** In this part, we combine the explicit integral formula of Shimura and the fact that the toric integrals can be  $p$ -adically interpolated to show the existence of a  $\Lambda$ -adic Shimura-Shintani-Waldspurger correspondence with the expected interpolation property. This follows very closely [21, §6].

Let  $\tilde{\kappa}_N \in \tilde{\mathcal{X}}_N^{\text{arith}}$  of signature  $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ . Let  $L_r$  denote the order of  $\mathbb{M}_2(F)$  consisting of matrices  $\begin{pmatrix} a & b/Mp^r \\ Mp^r c & d \end{pmatrix}$  with  $a, b, c, d \in \mathcal{O}_F$ . Define

$$\mathcal{O}_{B,r} := \iota^{-1}(i_F^{-1}(i_F(B \otimes 1) \cap L_r))$$

Then  $\mathcal{O}_{B,r}$  is the maximal order introduced in §2.1 (and denoted  $\mathcal{O}'_B$  there) defined in terms of the maximal order  $\mathcal{O}_B$  and the integer  $Mp^r$ . Also,  $S := \mathcal{O}_B \cap \mathcal{O}_{B,r}$  is an Eichler order of  $B$  of level  $Mp$  containing the fixed Eichler order  $R$  of level  $M$ . With  $\alpha \in V^* \cap \mathcal{O}_{B,1}$ , we have

$$(7) \quad i_F(\alpha) = \begin{pmatrix} a & b/(Mp) \\ c & -a \end{pmatrix}$$

in  $\mathbb{M}_2(F)$  with  $a, b, c \in \mathcal{O}_F$  and we can consider the quadratic forms

$$Q_\alpha(x, y) := cx^2 - 2axy - (b/(Mp))y^2,$$

and

$$(8) \quad \tilde{Q}_\alpha(x, y) := Mp \cdot Q_\alpha(x, y) = Mpcx^2 - 2Mpay - by^2.$$

Then  $\tilde{Q}_\alpha(x, y)$  has coefficients in  $\mathcal{O}_F$  and, composing with  $F \hookrightarrow \mathbb{R}$  and letting  $x = z$ ,  $y = 1$ , we recover  $Q_\alpha(z)$  and  $\tilde{Q}_\alpha(z)$  of §2.1 (defined by means of the isomorphism  $i_\infty$ ). Since each prime  $\ell|Mp$  is split in  $F$ , the elements  $a, b, c$  can be viewed as elements in  $\mathbb{Z}_\ell$  via our fixed embedding  $\mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$ , for any prime  $\ell|Mp$  (we will continue writing  $a, b, c$  for these elements, with a slight abuse of notation). So,

letting  $b_\alpha \in \mathbb{Z}$  such that  $i_\ell(\alpha) \equiv \left( \begin{smallmatrix} * & b_\alpha/(Mp) \\ * & * \end{smallmatrix} \right)$  modulo  $i_\ell(S \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$ , for all  $\ell|Mp$ , we have  $b \equiv b_\alpha$  modulo  $Mp\mathbb{Z}_\ell$  as elements in  $\mathbb{Z}_\ell$ , for all  $\ell|Mp$ , and thus we get

$$(9) \quad \eta_{\epsilon_{\tilde{\kappa}}}(\alpha) = \epsilon_{\tilde{\kappa}}(b_\alpha) = \epsilon_{\tilde{\kappa}}(b)$$

for  $b$  as in (7).

For any  $\nu \in \mathbb{D}$ , we may define an  $\mathcal{O}$ -valued measure  $j_\alpha(\nu)$  on  $\mathbb{Z}_p^\times$  by the formula:

$$\int_{\mathbb{Z}_p^\times} f(t) dj_\alpha(\nu)(t) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f(\tilde{Q}_\alpha(x, y)) d\nu(x, y).$$

for any continuous function  $f : \mathbb{Z}_p^\times \rightarrow \mathbb{C}_p$ . Recall that the group of  $\mathcal{O}$ -valued measures on  $\mathbb{Z}_p^\times$  is isomorphic to the Iwasawa algebra  $\mathcal{O}[[\mathbb{Z}_p^\times]]$ , and thus we may view  $j_\alpha(\nu)$  as an element in  $\mathcal{O}[[\mathbb{Z}_p^\times]]$  (see, for example, [1, §3.2]). In particular, for any group-like element  $[\lambda] \in \mathcal{O}[[\mathbb{Z}_p^\times]]$  we have:

$$\int_{\mathbb{Z}_p^\times} f(t) d([\lambda] \cdot j_\alpha(\nu))(t) = \int_{\mathbb{Z}_p^\times} \left( \int_{\mathbb{Z}_p^\times} f(ts) d[\lambda](s) \right) dj_\alpha(\nu)(t) = \int_{\mathbb{Z}_p^\times} f(\lambda t) dj_\alpha(\nu)(t).$$

On the other hand,

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f(\tilde{Q}_\alpha(x, y)) d(\lambda \cdot \nu) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f(\tilde{Q}_\alpha(\lambda x, \lambda y)) d\nu = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f(\lambda^2 \tilde{Q}_\alpha(x, y)) d\nu$$

and we conclude that  $j_\alpha(\lambda \cdot \nu) = [\lambda^2] \cdot j_\alpha(\nu)$ . In other words,  $j_\alpha$  is a  $\mathcal{O}[[\mathbb{Z}_p^\times]]$ -linear map

$$j_\alpha : \mathbb{D} \longrightarrow \mathcal{O}[[\mathbb{Z}_p^\times]]_\sigma.$$

Before going ahead, let us introduce some notation. Let  $\chi$  be a Dirichlet character modulo  $Mp^r$ , for a positive integer  $r$ , which we decompose accordingly with the isomorphism  $(\mathbb{Z}/Np^r\mathbb{Z})^\times \simeq (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p^r\mathbb{Z})^\times$  into the product  $\chi = \chi_N \cdot \chi_p$  with  $\chi_N : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  and  $\chi_p : (\mathbb{Z}/p^r\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ . Thus, we will write  $\chi(x) = \chi_N(x_N) \cdot \chi_p(x_p)$ , where  $x_N$  and  $x_p$  are the projections of  $x \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$  to  $(\mathbb{Z}/N\mathbb{Z})^\times$  and  $(\mathbb{Z}/p^r\mathbb{Z})^\times$ , respectively. To simplify the notation, we will suppress the  $N$  and  $p$  from the notation for  $x_N$  and  $x_p$ , thus simply writing  $x$  for any of the two. Using the isomorphism  $(\mathbb{Z}/N\mathbb{Z})^\times \simeq (\mathbb{Z}/M\mathbb{Z})^\times \times (\mathbb{Z}/D\mathbb{Z})^\times$ , decompose  $\chi_N$  as  $\chi_N = \chi_M \cdot \chi_D$  with  $\chi_M$  and  $\chi_D$  characters on  $(\mathbb{Z}/M\mathbb{Z})^\times$  and  $(\mathbb{Z}/D\mathbb{Z})^\times$ , respectively. In the following, we only need the case when  $\chi_D = 1$ .

Using the above notation, we may define a  $\mathcal{O}[[\mathbb{Z}_p^\times]]$ -linear map  $J_\alpha : \mathbb{D} \rightarrow \mathcal{O}[[\mathbb{Z}_p^\times]]$  by

$$J_\alpha(\nu) = \epsilon_{\tilde{\kappa}, M}(b) \cdot \epsilon_{\tilde{\kappa}, p}(-1) \cdot j_\alpha(\nu)$$

with  $b$  as in (7). Set  $\mathbb{D}_N := \mathbb{D} \otimes_{\mathcal{O}[[\mathbb{Z}_p^\times]]} \Lambda_N$ , where the map  $\mathcal{O}[[\mathbb{Z}_p^\times]] \rightarrow \Lambda_N$  is induced from the map  $\mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$  on group-like elements given by  $x \mapsto x \otimes 1$ . Then  $J_\alpha$  can be extended to a  $\Lambda_N$ -linear map  $J_\alpha : \mathbb{D}_N \rightarrow \Lambda_{N, \sigma}$ . Setting  $\mathbb{D}_{\mathcal{R}_N} := \mathcal{R}_N \otimes_{\Lambda_N} \mathbb{D}_N$  and extending by  $\mathcal{R}_N$ -linearity over  $\Lambda_N$  we finally obtain a  $\mathcal{R}_N$ -linear map, again denoted by the same symbol,

$$J_\alpha : \mathbb{D}_{\mathcal{R}_N} \longrightarrow \tilde{\mathcal{R}}_N.$$

For  $\nu \in \mathbb{D}_N$  and  $r \in \mathcal{R}_N$  we thus have

$$J_\alpha(r \otimes \nu) = \epsilon_{\tilde{\kappa}, M}(b) \cdot \epsilon_{\tilde{\kappa}, p}(-1) \cdot r \otimes j_\alpha(\nu).$$

For the next result, for any arithmetic point  $\kappa_N \in \mathcal{X}_N^{\text{arith}}$  coming from  $\kappa \in \mathcal{X}^{\text{arith}}$ , extend  $\rho_\kappa$  in (5) by  $\mathcal{R}_N$ -linearity over  $\mathcal{O}[\mathbb{Z}_p^\times]$ , to get a map

$$\rho_{\kappa_N} : \mathbb{D}_{\mathcal{R}_N} \longrightarrow V_{n_\kappa}$$

defined by  $\rho_{\kappa_N}(r \otimes \nu) := \rho_\kappa(\nu) \cdot \kappa_N(r)$ , for  $\nu \in \mathbb{D}$  and  $r \in \mathcal{R}_N$ . To simplify the notation, set

$$(10) \quad \langle \nu, \alpha \rangle_{\kappa_N} := \rho_{\kappa_N}(\nu)(\tilde{Q}_\alpha^{n_{\tilde{\kappa}}/2}).$$

The following is essentially [21, Lemma (6.1)].

**Lemma 3.2.** *Let  $\tilde{\kappa}_N \in \tilde{\mathcal{X}}_N^{\text{arith}}$  with signature  $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$  and define  $\kappa_N := p_N(\tilde{\kappa}_N)$ . Then for any  $\nu \in \mathbb{D}_{\mathcal{R}_N}$  we have:*

$$\tilde{\kappa}_N(J_\alpha(\nu)) = \eta_{\epsilon_{\tilde{\kappa}}}(\alpha) \cdot \langle \nu, \alpha \rangle_{\kappa_N}.$$

*Proof.* For  $\nu \in \mathbb{D}_N$  and  $r \in \mathcal{R}_N$  we have

$$\begin{aligned} \tilde{\kappa}_N(J_\alpha(r \otimes \nu)) &= \tilde{\kappa}_N(\epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot r \otimes j_\alpha(\nu)) \\ &= \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \tilde{\kappa}_N(r \otimes 1) \cdot \tilde{\kappa}_N(1 \otimes j_\alpha(\nu)) \\ &= \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \kappa_N(r) \cdot \int_{\mathbb{Z}_p^\times} \tilde{\kappa}_N(t) dj_\alpha(\nu) \end{aligned}$$

and thus, noticing that  $\tilde{\kappa}_N$  restricted to  $\mathbb{Z}_p^\times$  is  $\tilde{\kappa}_N(t) = \epsilon_{\tilde{\kappa},p}(t)t^{n_{\tilde{\kappa}}}$ , we have

$$\tilde{\kappa}_N(J_\alpha(r \otimes \nu)) = \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \kappa_N(r) \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon_{\tilde{\kappa},p}(\tilde{Q}_\alpha(x, y)) \tilde{Q}_\alpha(x, y)^{n_{\tilde{\kappa}}/2} d\nu.$$

Recalling (8), and viewing  $a, b, c$  as elements in  $\mathbb{Z}_p$ , we have, for  $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p^\times$ ,

$$\epsilon_{\tilde{\kappa},p}(\tilde{Q}_\alpha(x, y)) = \epsilon_{\tilde{\kappa},p}(-by^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\tilde{\kappa},p}(y^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\tilde{\kappa},p}^2(y) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\kappa,p}(y).$$

Thus, since  $\epsilon_{\tilde{\kappa}}(-1)^2 = 1$ , we get:

$$\tilde{\kappa}_N(J_\alpha(r \otimes \nu)) = \kappa_N(r) \cdot \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(b) \cdot \rho_\kappa(\nu)(\tilde{Q}_\alpha^{n_{\tilde{\kappa}}/2}) = \eta_{\epsilon_\kappa}(\alpha) \cdot \langle \nu, \alpha \rangle_{\kappa_N}$$

where for the last equality use (9) and (10).  $\square$

Define

$$\mathbb{W}_{\mathcal{R}_N} := \mathbb{W} \otimes_{\mathcal{O}[\mathbb{Z}_p^\times]} \mathcal{R}_N,$$

the structure of  $\mathcal{O}[\mathbb{Z}_p^\times]$ -module of  $\mathcal{R}_N$  being that induced by the composition of the two maps  $\mathcal{O}[\mathbb{Z}_p^\times] \rightarrow \Lambda_N \rightarrow \mathcal{R}_N$  described above. There is a canonical map

$$\vartheta : \mathbb{W}_{\mathcal{R}_N} \longrightarrow H^1(\Gamma_0, \mathbb{D}_{\mathcal{R}_N})$$

described as follows: if  $\nu_\gamma$  is a representative of an element  $\nu$  in  $\mathbb{W}$  and  $r \in \mathcal{R}_N$ , then  $\vartheta(\nu \otimes r)$  is represented by the cocycle  $\nu_\gamma \otimes r$ .

For  $\nu \in \mathbb{W}_{\mathcal{R}_N}$  represented by  $\nu_\gamma$  and  $\xi \geq 1$  an integer, define

$$\theta_\xi(\nu) := \sum_{C \in R(\Gamma_1), q(C)=\xi} \frac{J_{\alpha_C}(\nu_{\gamma_{\alpha_C}})}{t_{\alpha_C}}.$$

**Definition 3.3.** For  $\nu \in \mathbb{W}_{\mathcal{R}_N}$ , the formal Fourier expansion

$$\Theta(\nu) := \sum_{\xi \geq 1} \theta_\xi(\nu) q^\xi$$

in  $\mathcal{R}_N[[q]]$  is called the  $\Lambda$ -adic Shimura-Shintani-Waldspurger lift of  $\nu$ . For any  $\tilde{\kappa} \in \tilde{\mathcal{X}}^{\text{arith}}$ , the formal power series expansion

$$\Theta(\nu)(\tilde{\kappa}_N) := \sum_{\xi \geq 1} \tilde{\kappa}_N(\theta_\xi(\nu)) q^\xi$$

is called the  $\tilde{\kappa}$ -specialization of  $\Theta(\nu)$ .

There is a natural map

$$\mathbb{W}_{\mathcal{R}} \longrightarrow \mathbb{W}_{\mathcal{R}_N}$$

taking  $\nu \otimes r$  to itself (use that  $\mathcal{R}$  has a canonical map to  $\mathcal{R}_N \simeq \mathcal{R}[\Delta]$ , as described above). So, for any choice of sign,  $\Phi^\pm \in \mathbb{W}_{\mathcal{R}}$  will be viewed as an element in  $\mathbb{W}_{\mathcal{R}_N}$ .

From now on we will use the following notation. Fix  $\tilde{\kappa}_0 \in \tilde{\mathcal{X}}^{\text{arith}}$  and put  $\kappa_0 := p(\tilde{\kappa}_0) \in \mathcal{X}^{\text{arith}}$ . Recall the neighborhood  $\mathcal{U}_0$  of  $\kappa_0$  in Theorem 3.1. Define  $\tilde{\mathcal{U}}_0 := p^{-1}(\mathcal{U}_0)$  and

$$\tilde{\mathcal{U}}_0^{\text{arith}} := \tilde{\mathcal{U}}_0 \cap \tilde{\mathcal{X}}^{\text{arith}}.$$

For each  $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$  put  $\kappa = p(\tilde{\kappa}) \in \mathcal{U}_0^{\text{arith}}$ . Recall that if  $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$  is the signature of  $\tilde{\kappa}$ , then  $(\epsilon_{\kappa}, k_{\kappa}) := (\epsilon_{\tilde{\kappa}}^2, 2k_{\tilde{\kappa}})$  is that of  $\kappa_0$ . For any  $\kappa := p(\tilde{\kappa})$  as above, we may consider the modular form

$$f_{\kappa}^{\text{JL}} \in S_{k_{\kappa}}(\Gamma_{r_{\kappa}}, \epsilon_{\kappa})$$

and its Shimura-Shintani-Waldspurger lift

$$h_{\kappa} = \sum_{\xi} a_{\xi}(h_{\kappa}) q^{\xi} \in S_{k_{\kappa}+1/2}(4Np^{r_{\kappa}}, \chi_{\kappa}), \quad \text{where } \chi_{\kappa}(x) := \epsilon_{\tilde{\kappa}}(x) \left( \frac{-1}{x} \right)^{k_{\kappa}},$$

normalized as in (2) and (3). For our fixed  $\kappa_0$ , recall the elements  $\Phi := \Phi^+$  chosen as in Theorem 3.1 and define  $\phi_{\kappa} := \phi_{\kappa}^+$  and  $\Omega_{\kappa} := \Omega_{\kappa}^+$  for  $\kappa \in \mathcal{U}_0^{\text{arith}}$ .

**Proposition 3.4.** For all  $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$  such that  $r_{\kappa} = 1$  we have

$$\tilde{\kappa}_N(\theta_{\xi}(\Phi)) = \Omega_{\kappa} \cdot a_{\xi}(h_{\kappa}) \quad \text{and} \quad \Theta(\Phi)(\tilde{\kappa}_N) = \Omega_{\kappa} \cdot h_{\kappa}.$$

*Proof.* By Lemma 3.2 we have

$$\tilde{\kappa}_N(\theta_{\xi}(\Phi)) = \sum_{\mathcal{C} \in R(\Gamma_1), q(\mathcal{C})=\xi} \frac{\eta_{\epsilon_{\tilde{\kappa}}}(\alpha_{\mathcal{C}})}{\mathfrak{t}_{\alpha_{\mathcal{C}}}} \rho_{\kappa_N}(\Phi)(\tilde{Q}_{\alpha_{\mathcal{C}}}^{n_{\tilde{\kappa}}/2}).$$

Using Theorem 3.1, we get

$$\tilde{\kappa}_N(\theta_{\xi}(\Phi)) = \sum_{\mathcal{C} \in R(\Gamma_1), q(\mathcal{C})=\xi} \frac{\eta_{\epsilon_{\tilde{\kappa}}}(\alpha_{\mathcal{C}}) \cdot \Omega_{\kappa}}{\mathfrak{t}_{\alpha_{\mathcal{C}}}} \phi_{\kappa}(\tilde{Q}_{\alpha_{\mathcal{C}}}^{k_{\kappa}-1}).$$

Now (2) shows the statement on  $\tilde{\kappa}_N(\theta_{\xi}(\Phi))$ , while that on  $\Theta(\Phi)(\tilde{\kappa}_N)$  is a formal consequence of the previous one.  $\square$

**Corollary 3.5.** Let  $a_p$  denote the image of the Hecke operator  $T_p$  in  $\mathcal{R}$ . Then  $\Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi)$ .

*Proof.* For any  $\kappa \in \mathcal{X}^{\text{arith}}$ , let  $a_p(\kappa) := \kappa(T_p)$ , which is a  $p$ -adic unit by the ordinarity assumption. For all  $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$  with  $r_{\tilde{\kappa}} = 1$ , we have

$$\Theta(\Phi)(\tilde{\kappa}_N)|T_p^2 = \Omega_{\tilde{\kappa}} \cdot h_{\tilde{\kappa}}|T_p^2 = a_p(\kappa) \cdot \Omega_{\kappa} \cdot h_{\kappa} = a_p(\kappa) \cdot \Theta(\Phi)(\tilde{\kappa}_N).$$

Consequently,

$$\tilde{\kappa}_N(\theta_{\xi p^2}(\Phi)) = a_p(\kappa) \cdot \tilde{\kappa}_N(\theta_{\xi}(\Phi))$$

for all  $\tilde{\kappa}$  such that  $r_{\tilde{\kappa}} = 1$ . Since this subset is dense in  $\tilde{\mathcal{X}}_N$ , we conclude that  $\theta_{\xi p^2}(\Phi) = a_p \cdot \theta_{\xi}(\Phi)$  and so  $\Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi)$ .  $\square$

For any integer  $n \geq 1$  and any quadratic form  $Q$  with coefficients in  $F$ , write  $[Q]_n$  for the class of  $Q$  modulo the action of  $i_F(\Gamma_n)$ . Define  $\mathcal{F}_{n,\xi}$  to be the subset of the  $F$ -vector space of quadratic forms with coefficients in  $F$  consisting of quadratic forms  $\tilde{Q}_{\alpha}$  such that  $\alpha \in V^* \cap \mathcal{O}_{B,n}$  and  $-\text{nr}(\alpha) = \xi$ . Writing  $\delta_{\tilde{Q}_{\alpha}}$  for the discriminant of  $Q_{\alpha}$ , the above set can be equivalently described as

$$\mathcal{F}_{n,\xi} := \{\tilde{Q}_{\alpha} | \alpha \in V^* \cap \mathcal{O}_{B,n}, \delta_{\tilde{Q}_{\alpha}} = Np^n \xi\}.$$

Define  $\mathcal{F}_{n,\xi}/\Gamma_n$  to be the set  $\{[\tilde{Q}_{\alpha}]_n | \tilde{Q}_{\alpha} \in \mathcal{F}_{n,\xi}\}$  of equivalence classes of  $\mathcal{F}_{n,\xi}$  under the action of  $i_F(\Gamma_n)$ . A simple computation shows that  $Q_{g^{-1}\alpha g} = Q_{\alpha}|g$  for all  $\alpha \in V^*$  and all  $g \in \Gamma_n$ , and thus we find

$$\mathcal{F}_{n,\xi}/\Gamma_n = \{[\tilde{Q}_{\alpha}]_n | \alpha \in V^* \cap \mathcal{O}_{B,n}, \delta_{\tilde{Q}_{\alpha}} = Np^n \xi\}.$$

We also note that, in the notation of §2.1, if  $f$  has weight character  $\psi$ , defined modulo  $Np^n$ , and level  $\Gamma_n$ , the Fourier coefficients  $a_{\xi}(h)$  of the Shimura-Shintani-Waldspurger lift  $h$  of  $f$  are given by

$$(11) \quad a_{\xi}(h) = \sum_{[Q] \in \mathcal{F}_{n,\xi}/\Gamma_n} \frac{\psi(Q)}{\mathfrak{t}_Q} \phi_f^+(Q(z)^{k-1})$$

and, if  $Q = \tilde{Q}_{\alpha}$ , we put  $\psi(Q) := \eta_{\psi}(b_{\alpha})$  and  $\mathfrak{t}_Q := \mathfrak{t}_{\alpha}$ . Also, if we let

$$\mathcal{F}_n/\Gamma_n := \coprod_{\xi} \mathcal{F}_{n,\xi}/\Gamma_n$$

we can write

$$(12) \quad h = \sum_{[Q] \in \mathcal{F}_n/\Gamma_n} \frac{\psi(Q)}{\mathfrak{t}_Q} \phi_f^+(Q(z)^{k-1}) q^{\delta_Q/(Np^n)}.$$

Fix now an integer  $m \geq 1$  and let  $n \in \{1, m\}$ . For any  $t \in (\mathbb{Z}/p^n\mathbb{Z})^{\times}$  and any integer  $\xi \geq 1$ , define  $\mathcal{F}_{m,\xi,t}$  to be the subset of  $\mathcal{F}_{n,\xi}$  consisting of forms such that  $Np^n b_{\alpha} \equiv t \pmod{Np^m}$ . Also, define  $\mathcal{F}_{m,\xi,t}/\Gamma_n$  to be the set of equivalence classes of  $\mathcal{F}_{m,\xi,t}$  under the action of  $i_F(\Gamma_n)$ . If  $\alpha \in V^* \cap \mathcal{O}_{B,m}$  and  $i_F(\alpha) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ , then

$$(13) \quad \tilde{Q}_{\alpha}(x, y) = Np^n cx^2 - 2Np^n axy - Np^n by^2$$

from which we see that there is an inclusion  $\mathcal{F}_{m,\xi,t} \subseteq \mathcal{F}_{1,\xi p^{m-1},t}$ . If  $\tilde{Q}_{\alpha}$  and  $\tilde{Q}_{\alpha'}$  belong to  $\mathcal{F}_{m,\xi,t}$ , and  $\alpha' = g\alpha g^{-1}$  for some  $g \in \Gamma_m$ , then, since  $\Gamma_m \subseteq \Gamma_1$ , we see that  $\tilde{Q}_{\alpha}$  and  $\tilde{Q}_{\alpha'}$  represent the same class in  $\mathcal{F}_{1,\xi p^{m-1},t}/\Gamma_1$ . This shows that  $[\tilde{Q}_{\alpha}]_m \mapsto [\tilde{Q}_{\alpha}]_1$  gives a well-defined map

$$\pi_{m,\xi,t} : \mathcal{F}_{m,\xi,t}/\Gamma_m \longrightarrow \mathcal{F}_{1,\xi p^{m-1},t}/\Gamma_1.$$

**Lemma 3.6.** *The map  $\pi_{m,\xi,t}$  is bijective.*

*Proof.* We first show the injectivity. For this, suppose  $\tilde{Q}_\alpha$  and  $\tilde{Q}_{\alpha'}$  are in  $\mathcal{F}_{m,\xi,t}$  and  $[\tilde{Q}_\alpha]_1 = [\tilde{Q}_{\alpha'}]_1$ . So there exists  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  in  $i_F(\Gamma_1)$  such that  $\tilde{Q}_\alpha = \tilde{Q}_{\alpha'}|g$ . If  $\tilde{Q}_\alpha = cx^2 - 2axy - by^2$ , and easy computation shows that  $\tilde{Q}_{\alpha'} = c'x^2 - 2a'xy - b'y^2$  with

$$\begin{aligned} c' &= c\alpha^2 - 2a\alpha\gamma - b\gamma^2 \\ a' &= -c\alpha\beta + a\beta\gamma + a\alpha\delta + b\gamma\delta \\ b' &= -c\beta^2 + 2a\beta\delta + b\delta^2. \end{aligned}$$

The first condition shows that  $\gamma \equiv 0 \pmod{Np^m}$ . We have  $b \equiv b' \equiv t \pmod{Np^m}$ , so  $\delta^2 \equiv 1 \pmod{Np^m}$ . Since  $\delta \equiv 1 \pmod{Np}$ , we see that  $\delta \equiv 1 \pmod{Np^m}$  too.

We now show the surjectivity. For this, fix  $[\tilde{Q}_{\alpha_c}]_1$  in the target of  $\pi$ , and choose a representative

$$\tilde{Q}_{\alpha_c} = cx^2 - 2axy - by^2$$

(recall  $Mp^m\xi|\delta_{\tilde{Q}_{\alpha_c}}$ ,  $Np|c$ ,  $Np|a$ , and  $b \in \mathcal{O}_F^\times$ , the last condition due to  $\eta_\psi(\alpha_c) \neq 0$ ). By the Strong Approximation Theorem, we can find  $\tilde{g} \in \Gamma_1$  such that

$$i_\ell(\tilde{g}) \equiv \begin{pmatrix} 1 & 0 \\ ab^{-1} & 1 \end{pmatrix} \pmod{Np^m}$$

for all  $\ell|Np$ . Take  $g := i_F(\tilde{g})$ , and put  $\alpha := g^{-1}\alpha_c g$ . An easy computation, using the expressions for  $a', b', c'$  in terms of  $a, b, c$  and  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  as above, shows that  $\alpha \in V^* \cap \mathcal{O}_{B,m}$ ,  $\eta_\psi(\alpha) = t$  and  $\delta_{\tilde{Q}_\alpha} = Np^m\xi$ , and it follows that  $\tilde{Q}_\alpha \in \mathcal{F}_{m,\xi,t}$ . Now

$$\pi([\tilde{Q}_\alpha]_m) = [\tilde{Q}_\alpha]_1 = [\tilde{Q}_{g^{-1}\alpha_c g}]_1 = [\tilde{Q}_{\alpha_c}]_1$$

where the last equality follows because  $g \in \Gamma_1$ .  $\square$

**Proposition 3.7.** *For all  $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$  we have*

$$\Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa-1} = \Omega_\kappa \cdot h_\kappa.$$

*Proof.* For  $r_\kappa = 1$ , this is Proposition 3.4 above, so we may assume  $r_\kappa \geq 2$ . As in the proof of Proposition 3.4, combining Lemma 3.2 and Theorem 3.1 we get

$$\Theta(\Phi)(\tilde{\kappa}_N) = \sum_{\xi \geq 1} \left( \sum_{\mathcal{C} \in R(\Gamma_1), q(\mathcal{C})=\xi} \frac{\eta_{\tilde{\kappa}}(\alpha_{\mathcal{C}}) \cdot \Omega_\kappa}{\mathfrak{t}_{\alpha_{\mathcal{C}}}} \phi_\kappa(\tilde{Q}_{\alpha_{\mathcal{C}}}^{k_\kappa-1}) \right) q^\xi$$

which, by (11) and (12) above we may rewrite as

$$\Theta(\Phi)(\tilde{\kappa}_N) = \sum_{[Q] \in \mathcal{F}_1/\Gamma_1} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathfrak{t}_Q} \phi_\kappa(Q^{k_\kappa-1}) q^{\delta_Q/(Np)}$$

By definition of the action of  $T_p$  on power series, we have

$$\Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa-1} = \sum_{[Q] \in \mathcal{F}_1/\Gamma_1, p^{r_\kappa}|\delta_Q} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathfrak{t}_Q} \phi_\kappa(Q^{k_\kappa-1}) q^{\delta_Q/(Np^{r_\kappa})}.$$

Setting  $\mathcal{F}_{n,t}/\Gamma_n := \coprod_{\xi \geq 1} \mathcal{F}_{n,t,\xi}/\Gamma_n$  for  $n \in \{1, r_\kappa\}$ , Lemma 3.6 shows that

$$\mathcal{F}_{1,t}^* := \{[Q] \in \mathcal{F}_{1,t}/\Gamma_{1,t} \text{ such that } p^{r_\kappa}|\delta_Q\} \text{ is equal to } \mathcal{F}_{r_\kappa,t}.$$



Therefore, splitting the above sum over  $t \in (\mathbb{Z}/Np^{r_\kappa}\mathbb{Z})^\times$ , we get

$$\begin{aligned} \Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa-1} &= \sum_{t \in (\mathbb{Z}/p^{r_\kappa-1}\mathbb{Z})^\times} \sum_{[Q] \in \mathcal{F}_{1,t}^*} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{t_Q} \phi_\kappa(Q^{k_\kappa-1}) q^{\delta_Q/(Np^{r_\kappa})} \\ &= \sum_{t \in (\mathbb{Z}/p^{r_\kappa-1}\mathbb{Z})^\times} \sum_{[Q] \in \mathcal{F}_{m,t}/\Gamma_m} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{t_Q} \phi_\kappa(Q^{k_\kappa-1}) q^{\delta_Q/(Np^{r_\kappa})} \\ &= \sum_{[Q] \in \mathcal{F}_m/\Gamma_m} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{t_Q} \phi_\kappa(Q^{k_\kappa-1}) q^{\delta_Q/(Np^{r_\kappa})}. \end{aligned}$$

Comparing this expression with (12) gives the result.  $\square$

We are now ready to state the analogue of [21, Thm. 3.3], which is our main result. For the reader's convenience, we briefly recall the notation appearing below. We denote by  $\mathcal{X}$  the points of the ordinary Hida Hecke algebra, and by  $\mathcal{X}^{\text{arith}}$  its arithmetic points. For  $\kappa_0 \in \mathcal{X}^{\text{arith}}$ , we denote by  $\mathcal{U}_0$  the  $p$ -adic neighborhood of  $\kappa_0$  appearing in the statement of Theorem 3.1 and put  $\mathcal{U}_0^{\text{arith}} := \mathcal{U}_0 \cap \mathcal{X}^{\text{arith}}$ . We also denote by  $\Phi = \Phi^+ \in \mathbb{W}_{\mathcal{R}}^{\text{ord}}$  the cohomology class appearing in Theorem 3.1. The points  $\tilde{\mathcal{X}}$  of the metaplectic Hida Hecke algebra defined in §3.3 are equipped with a canonical map  $p : \tilde{\mathcal{X}}^{\text{arith}} \rightarrow \mathcal{X}^{\text{arith}}$  on arithmetic points. Let  $\tilde{\mathcal{U}}_0^{\text{arith}} := \tilde{\mathcal{U}}_0 \cap \tilde{\mathcal{X}}^{\text{arith}}$ . For each  $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$  put  $\kappa = p(\tilde{\kappa}) \in \mathcal{U}_0^{\text{arith}}$ . Recall that if  $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$  is the signature of  $\tilde{\kappa}$ , then  $(\epsilon_\kappa, k_\kappa) := (\epsilon_{\tilde{\kappa}}^2, 2k_{\tilde{\kappa}})$  is that of  $\kappa_0$ . For any  $\kappa := p(\tilde{\kappa})$  as above, we may consider the modular form

$$f_\kappa^{\text{JL}} \in S_{k_\kappa}(\Gamma_{r_\kappa}, \epsilon_\kappa)$$

and its Shimura-Shintani-Waldspurger lift

$$h_\kappa = \sum_{\xi} a_\xi(h_\kappa) q^\xi \in S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi_\kappa), \quad \text{where } \chi_\kappa(x) := \epsilon_{\tilde{\kappa}}(x) \left( \frac{-1}{x} \right)^{k_\kappa},$$

normalized as in (2) and (3). Finally, for  $\tilde{\kappa} \in \tilde{\mathcal{X}}^{\text{arith}}$ , we denote by  $\tilde{\kappa}_N$  its extension to the metaplectic Hecke algebra  $\tilde{\mathcal{R}}_N$  defined in §3.3.

**Theorem 3.8.** *Let  $\kappa_0 \in \mathcal{X}^{\text{arith}}$ . Then there exists a choice of  $p$ -adic periods  $\Omega_\kappa$  for  $\kappa \in \mathcal{U}_0$  such that the  $\Lambda$ -adic Shimura-Shintani-Waldspurger lift of  $\Phi$*

$$\Theta(\Phi) := \sum_{\xi \geq 1} \theta_\xi(\Phi) q^\xi$$

in  $\mathcal{R}_N[[q]]$  has the following properties:

- (1)  $\Omega_{\kappa_0} \neq 0$ .
- (2) For any  $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$ , the  $\tilde{\kappa}$ -specialization of  $\Theta(\Phi)$

$$\Theta(\nu)(\tilde{\kappa}_N) := \sum_{\xi \geq 1} \tilde{\kappa}(\theta_\xi(\Phi)) q^\xi \text{ belongs to } S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi'_\kappa),$$

where  $\chi'_\kappa(x) := \chi_\kappa(x) \cdot \left( \frac{p}{x} \right)^{k_\kappa-1}$ , and satisfies

$$\Theta(\Phi)(\tilde{\kappa}_N) = \Omega_\kappa \cdot h_\kappa|T_p^{1-r_\kappa}.$$

*Proof.* The elements  $\Omega_\kappa$  are those  $\Omega_\kappa^+$  appearing in Theorem 3.1, which we used in Propositions 3.4 and 3.7 above, so (1) is clear. Applying  $T_p^{r_\kappa-1}$  to the formula of Proposition 3.7, using Corollary 3.5 and applying  $a_p(\kappa)^{1-r_\kappa}$  on both sides gives

$$\Theta(\Phi)(\tilde{\kappa}_N) = a_p(\kappa)^{1-r_\kappa} \Omega_\kappa \cdot h_\kappa |T_p^{r_\kappa-1}.$$

By [18, Prop. 1.9], each application of  $T_p$  has the effect of multiplying the character by  $\left(\frac{p}{\cdot}\right)$ , hence

$$h'_\kappa := h_\kappa |T_p^{r_\kappa-1} \in S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi'_\kappa)$$

with  $\chi'_\kappa$  as in the statement. This gives the first part of (2), while the last formula follows immediately from Proposition 3.7.  $\square$

*Remark 3.9.* Theorem 1.1 is a direct consequence of Theorem 3.8, as we briefly show below.

Recall the embedding  $\mathbb{Z}^{\geq 2} \hookrightarrow \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$  which sends  $k \in \mathbb{Z}^{\geq 2}$  to the character  $x \mapsto x^{k-2}$ . Extending characters by  $\mathcal{O}$ -linearity gives a map

$$\mathbb{Z}^{\geq 2} \hookrightarrow \mathcal{X}(\Lambda) := \text{Hom}_{\mathcal{O}\text{-alg}}^{\text{cont}}(\Lambda, \bar{\mathbb{Q}}_p).$$

We denote by  $k^{(\Lambda)}$  the image of  $k \in \mathbb{Z}^{\geq 2}$  in  $\mathcal{X}(\Lambda)$  via this embedding. We also denote by  $\varpi : \mathcal{X} \rightarrow \mathcal{X}(\Lambda)$  the finite-to-one map obtained by restriction of homomorphisms to  $\Lambda$ . Let  $k^{(\mathcal{R})}$  be a point in  $\mathcal{X}$  of signature  $(k, 1)$  such that  $\varpi(k^{(\mathcal{R})}) = k^{(\Lambda)}$ . A well-known result by Hida (see [6, Cor. 1.4]) shows that  $\mathcal{R}/\Lambda$  is unramified at  $k^{(\Lambda)}$ . As shown in [21, §1], this implies that there is a section  $s_{k^{(\Lambda)}}$  of  $\varpi$  which is defined on a neighborhood  $\mathcal{U}_{k^{(\Lambda)}}$  of  $k^{(\Lambda)}$  in  $\mathcal{X}(\Lambda)$  and sends  $k^{(\Lambda)}$  to  $k^{(\mathcal{R})}$ .

Fix now  $k_0$  as in the statement of Theorem 1.1, corresponding to a cuspform  $f_0$  of weight  $k_0$  with trivial character. The form  $f_0$  corresponds to an arithmetic character  $k_0^{(\mathcal{R})}$  of signature  $(1, k_0)$  belonging to  $\mathcal{X}$ . Let  $\mathcal{U}'_0$  be the inverse image of  $\mathcal{U}_0$  under the section  $s_{k_0^{(\Lambda)}}^{-1}$ , where  $\mathcal{U}_0 \subseteq \mathcal{X}$  is the neighborhood of  $k_0^{(\mathcal{R})}$  in Theorem 3.8. Extending scalars by  $\mathcal{O}$  gives, as above, an injective continuous map  $\text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \hookrightarrow \mathcal{X}(\Lambda)$ , and we let  $U_0$  be any neighborhood of the character  $x \mapsto x^{k_0-2}$  which maps to  $\mathcal{U}'_0$  and is contained in the residue class of  $k_0$  modulo  $p-1$ . Composing this map with the section  $\mathcal{U}'_0 \hookrightarrow \mathcal{U}_0$  gives a continuous injective map

$$\varsigma : U_0 \hookrightarrow \mathcal{U}'_0 \hookrightarrow \mathcal{U}_0$$

which takes  $k_0$  to  $k_0^{(\mathcal{R})}$ , since by construction the image of  $k_0$  by the first map is  $k_0^{(\Lambda)}$ . We also note that, more generally,  $\varsigma(k) = k^{(\mathcal{R})}$  because by construction  $\varsigma(k)$  restricts to  $k^{(\Lambda)}$  and its signature is  $(1, k)$ , since the character of  $\varsigma(k)$  is trivial. To show the last assertion, recall that the character of  $\varsigma(k)$  is  $\psi_k \cdot \psi_{\mathcal{R}} \cdot \omega^{-k}$ , and note that  $\psi_k$  is trivial because  $k^{(\Lambda)}(x) = x^{k-1}$ , and  $\psi_{\mathcal{R}} \cdot \omega^{-k}$  is trivial because the same is true for  $k_0$  and  $k \equiv k_0$  modulo  $p-1$ . In other words, arithmetic points in  $\varsigma(U_0)$  correspond to cuspforms with trivial character. This is the Hida family of forms with trivial character that we considered in the Introduction.

We can now prove Theorem 1.1. For all  $k \in U_0 \cap \mathbb{Z}^{\geq 2}$ , put  $\Omega_k := \Omega_{k^{(\mathcal{R})}}$  and define  $\Theta := \Theta(\Phi) \circ \varsigma$  with  $\Phi$  as in Theorem 3.8 for  $\kappa_0 = k_0^{(\mathcal{R})}$ . Applying Theorem 3.8 to  $k_0^{(\mathcal{R})}$ , and restricting to  $\varsigma(U_0)$ , shows that  $U_0$ ,  $\Omega_k$  and  $\Theta$  satisfy the conclusion of Theorem 1.1.

*Remark 3.10.* For  $\tilde{\kappa} \in \tilde{\mathcal{U}}_0^{\text{arith}}$  of signature  $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$  with  $r_{\tilde{\kappa}} = 1$  as in the above theorem,  $h_{\tilde{\kappa}}$  is trivial if  $(-1)^{k_{\tilde{\kappa}}} \neq \epsilon_{\tilde{\kappa}}(-1)$ . However, since  $\phi_{\kappa_0} \neq 0$ , it follows that  $h_{\kappa_0}$  is not trivial as long as the necessary condition  $(-1)^{k_0} = \epsilon_0(-1)$  is verified.

*Remark 3.11.* This result can be used to produce a quaternionic  $\Lambda$ -adic version of the Saito-Kurokawa lifting, following closely the arguments in [8, Cor. 1].

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